A geocentric relativistic VLBI–Model

Klaus Börger, Geodetic Institute, University of Bonn, Nussallee 17, 53115 Bonn
e–mail: boerger@uni–bonn.de

February 1999

Abstract:
Three relativistic models are available in the data analysis software CALC/SOLVE, that is to say the Shapiro–, the Hellings– and the Consensus–Model. They all have in common the same starting point, i.e. the motion of a photon is described in a barycentric spacetime and only the subsequent transformation into a geocentric spacetime finally provides the relation between observation and model–quantities. Further these models only supply an observation equation, but an adequate stellar model for the radio sources is missing. Based on this a geocentric modelling with a relativistic stellar model is presented. In order to investigate the new model, different analyses were made, which finally confirm the general validity of the geocentric relativistic model. Comparing parameter estimations shows that the new model actually yields improved results.

1 Introduction

At present three relativistic VLBI–models are implemented in the data analysis software CALC–SOLVE. These models – considered as a series – reflect the continuos development and improvement of theory and practice. Specified there are the Shapiro–Model, which is the original relativistic VLBI–Model, further the Hellings–Model as an enlargement and adaption to improved observation accuracies and finally the Consensus–Model, which is the standard relativistic VLBI–model, recommended for use by the IERS. As a matter of principle there is no substantial difference between these models, although they are different in derivation [*Eubanks (1991)*]. These models all have in common a primary barycentric reference and they all do not have a stellar model, i.e. a general relativistic transformation from the celestial system to the geocentric system. Starting from these states, a new approach, especially regarding geodetic aspects, is opposed to the customary models.

2 Intentions and features of the new model

The purpose is to develop an **alternative approach**, which does not only supply an observation equation, but also demonstrates a method, which shows new aspects and perspectives by a comprehensive consideration of the matter. The new model is characterized by the inclusion of a stellar model and a **geocentric approach**. Concerning this, at a first glance it is true, that both procedures, the geocentric one as well as the barycentric one, are equivalent, but from a geodetic point of view a geocentric modelling is better qualified, because all observation and estimation parameters are related to earth, except the radio source positions. Further relativistic effects w.r.t. the earth are smaller than those related to the sun. And a final argument for a geocentric modelling is the general possibility of transfering the results to other space geodetic methods which have the delay as principal observable, too. In order to connect theory and experiment data analysis using the new model is necessary. The performed investigations mainly serve two aspects, that is to say the verification and classification of the quality of the new model.

3 Modelling

Four aspects are significant for modelling, which are shown in figure 1. At the telescopes a signal–stream is observed and a time information is assigned. In relativity one distinguishes between coordinate time and proper time, which for example is realized by a clock on the surface of the Earth, i.e. each station clock provides a proper time. Therefore an exact **definition of the observable**, which is the difference between two points of time, is absolutely necessary. From this follows the proper time of arrival is the **primary measuring quantity**

$$\tau_a \textit{ in station } a \qquad\qquad \tau_b \textit{ in station } b\ . \tag{3–1a}$$

It is not possible to synchronize different positioned clocks, for this purpose one has to compare the corresponding coordinate times. Assumed a negligible station velocity and ignoring a clock correction the coordinate times belonging to (3–1a) are determined by

$$t_a = (g_{00})_a^{-\frac{1}{2}}\,\tau_a\ , \qquad\qquad t_b = (g_{00})_b^{-\frac{1}{2}}\,\tau_b\ . \tag{3–1b}$$

g_{00} is the time component of the metric tensor $g_{\alpha\beta}$. With this preliminary the **observable of the geodetic VLBI** is obtained from the difference of the quantities in (3-1b) [see *Börger (1998)*]

$$\begin{aligned} \Delta t &= t_b - t_a \\ &= \Delta\tau - \left\{ [1-(g_{00})_b^{-\frac{1}{2}}]\,\tau_b - [1-(g_{00})_a^{-\frac{1}{2}}]\,\tau_a \right\} \qquad (3\text{–}2a) \\ &= \Delta\tau - \left\{ [(g_{00})_a^{-\frac{1}{2}} - (g_{00})_b^{-\frac{1}{2}}]\,\tau_a + [1-(g_{00})_b^{-\frac{1}{2}}]\,\Delta\tau \right\}\ , \\ \Delta\tau &= \tau_b - \tau_a\ . \qquad (3\text{–}2b) \end{aligned}$$

The first expression in the brace on the right side of the last equation (3–2a) is the **difference in geometry** and the second one is the classical **relation between coordinate– and proper time**.

Further the introduction of different systems is necessary, because parameters and vectors are presented in different systems. I.e. in the language of relativity one has to **define different spacetimes**, and it is distinguished between three spacetimes, that is to say the catalog–spacetime $\hat{\Sigma}$ in which the radio source positions are tabled; the geocentric spacetime Σ, which is the basis of this approach and local spacetimes $\overline{\Sigma}$, which are connected to the telescopes. In addition to this one has to determine the relations between these spacetimes, i.e. one has to determine the transformations. For this purpose the concept of tetrads is applied [see *Börger (1998)*].

Finally the **determination of the signal–path** as well as the **determination of the signal–velocity** are needed. Concerning the last aspect one has to deal with electrodynamics. The determination of the signal–path has to be explained in more detail. The signal–path depends on geometry, because geometry defines geodetic lines and therewith those lines on which electromagnetic signals travel. Within the Einstein theory of gravitation the determination of geometry results from the solution of the field equations. These field equations require an adequate model of matter, which should contain all forms of energy and momentum as shown in figure 1. This of course is an interesting methodical aspect. The **solution of the Einstein's field equations**, i.e. the determination of expressions for the metric coefficients is described in e.g. *Börger (1996).*

The next step is to express all these ideas and thoughts in mathematical terms in order to get a model and an observation equation. A detailed description of the relations and the derivation of the basic equations is given in *Börger (1998).*

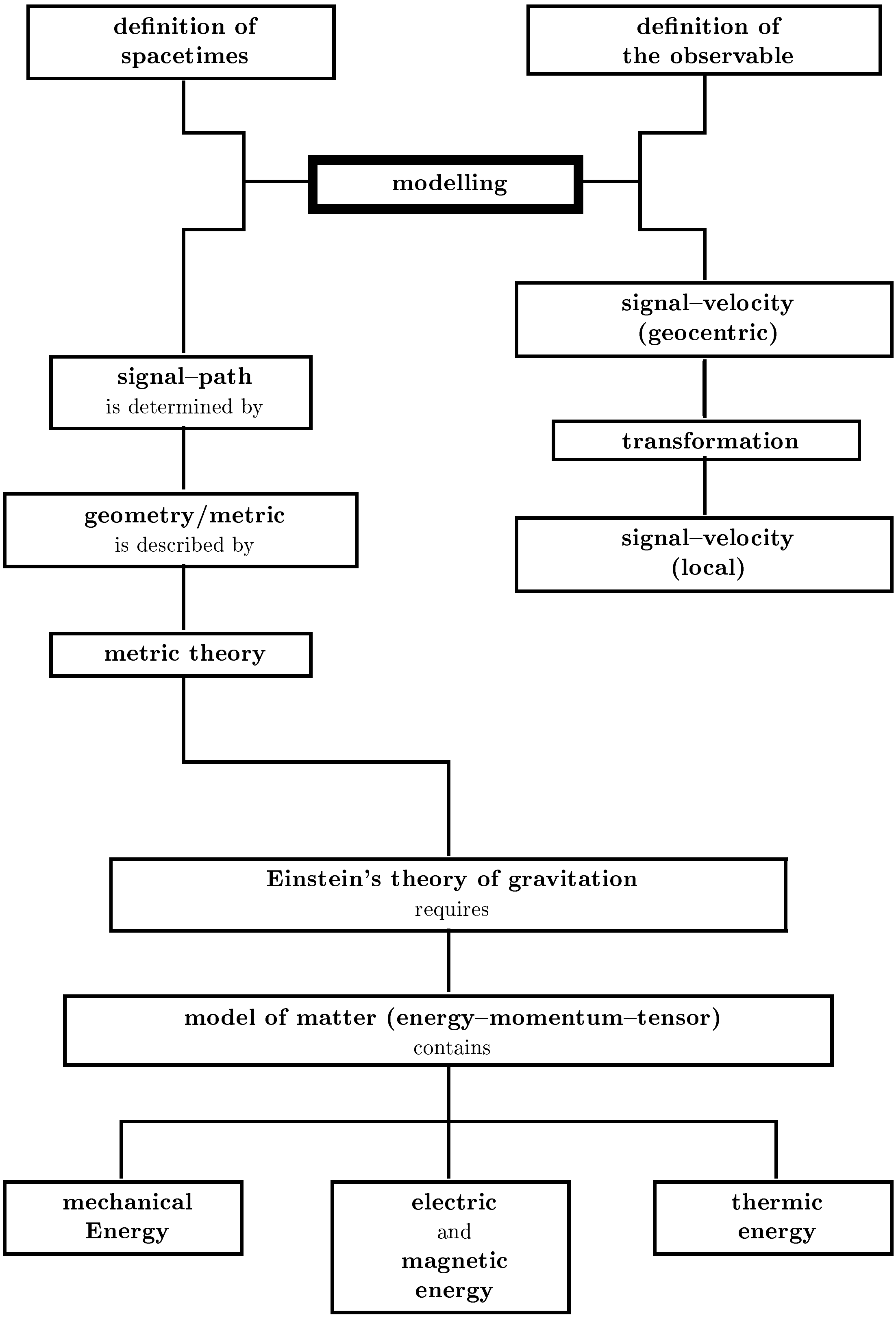

Fig. 1: procedure of modelling

Due to *Börger (1998) (3.6–6)* the **geocentric relativistic observation equation of the geodetic VLBI** reads:

$$\Delta t \;=\; \frac{s}{v_H}\left\{1+\frac{1}{v_H^2}\sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\frac{1}{n!\,m!}\,(\Delta_{ij}\,D_n^i\,D_m^j)\,\Delta t^{(n+m-2)}\right\}^{-\frac{1}{2}}\;, \tag{3–3a}$$

an expression, which determines the delay in an iterative mode. The first term on the right side is a first approximation, the so–called **Newtonian delay**. Up to the order $n = 4$, the D_n^i are determined as follows:

$$D_1^i \;=\; \left(\frac{dx_H^i}{dt}\right)_b = c\left(\frac{k^i}{k^0}\right)_b = v_{Hb}^i \tag{3–3b}$$

$$\begin{aligned} D_2^i \;&=\; \left(\frac{d^2\,x^i}{dt^2}\right)_b \\ &=\; \left(\frac{1}{c}\,\Gamma^0_{\beta\gamma}\,v^i_{Hb}-\Gamma^i_{\beta\gamma}\right)_b\,v^\beta_{Hb}\,v^\gamma_{Hb}\;+\;\left(P^i-\frac{1}{c}\,P^0\,v^i_{Hb}\right)_b \end{aligned} \tag{3–3c}$$

$$\begin{aligned} D_3^i \;=\;& \Big\{\Big(\frac{1}{c}\,\Gamma^0_{\alpha\beta,\gamma}\;+\;\frac{3}{c^2}\,\Gamma^0_{\alpha\beta}\,\Gamma^0_{\gamma\delta}\,v^\delta_H\;-\;\frac{2}{c}\,\Gamma^0_{\alpha\mu}\,\Gamma^\mu_{\beta\gamma}\Big)\,v^i_H \\ &-\Big(\Gamma^i_{\alpha\beta,\gamma}\;-\;2\,\Gamma^i_{\alpha\mu}\,\Gamma^\mu_{\beta\gamma}\;+\;\frac{3}{c}\,\Gamma^0_{\alpha\beta}\,\Gamma^i_{\gamma\delta}\,v^\delta_H\Big)\Big\}_b\,v^\alpha_{Hb}\,v^\beta_{Hb}\,v^\gamma_{Hb} \end{aligned} \tag{3–3d}$$

$$\begin{aligned} D_4^i \;=\;& \Big\{\Big(\frac{1}{c}\,\Gamma^0_{\alpha\beta,\gamma\delta}\;-\;\frac{2}{c}\Gamma^0_{\alpha\mu,\delta}\,\Gamma^\mu_{\beta\gamma}\;-\;\frac{2}{c}\Gamma^0_{\alpha\mu}\,\Gamma^\mu_{\beta\gamma,\delta} \\ &-\frac{2}{c}\Gamma^0_{\epsilon\alpha,\beta}\,\Gamma^\epsilon_{\gamma\delta}\;-\;\frac{1}{c}\Gamma^0_{\alpha\beta,\mu}\,\Gamma^\mu_{\gamma\delta}\;+\;\frac{2}{c}\Gamma^0_{\mu\nu}\,\Gamma^\nu_{\alpha\beta}\,\Gamma^\mu_{\gamma\delta} \\ &+\frac{4}{c}\Gamma^0_{\alpha\mu}\,\Gamma^\mu_{\nu\beta}\,\Gamma^\nu_{\gamma\delta}\;+\;\frac{10}{c^2}\,\Gamma^0_{\alpha\beta}\,\Gamma^0_{\gamma\delta,\nu}\,v^\nu_{Hb} \\ &-\frac{20}{c^2}\,\Gamma^0_{\alpha\beta}\,\Gamma^0_{\mu\gamma}\,\Gamma^\mu_{\nu\delta}\,v^\nu_{Hb}\;+\;\frac{15}{c^3}\,\Gamma^0_{\alpha\beta}\,\Gamma^0_{\gamma\delta}\,\Gamma^0_{\mu\nu}\,v^\mu_{Hb}\,v^\nu_{Hb}\Big)\,v^i_{Hb} \\ &-\Gamma^i_{\alpha\beta,\gamma\delta}\;+\;2\,\Gamma^i_{\alpha\mu,\delta}\,\Gamma^\mu_{\beta\gamma}\;+\;2\,\Gamma^i_{\alpha\mu}\,\Gamma^\mu_{\beta\gamma,\delta} \\ &+2\,\Gamma^i_{\mu\alpha,\beta}\,\Gamma^\mu_{\gamma\delta}\;+\;\Gamma^i_{\alpha\beta,\mu}\,\Gamma^\mu_{\gamma\delta}\;-\;2\,\Gamma^i_{\mu\nu}\,\Gamma^\mu_{\alpha\beta}\,\Gamma^\nu_{\gamma\delta} \\ &-4\,\Gamma^i_{\alpha\mu},\Gamma^\mu_{\nu\beta}\,\Gamma^\nu_{\gamma\delta}\;-\;\frac{4}{c}\,\Gamma^0_{\alpha\beta,\gamma}\,\Gamma^i_{\delta\mu}\,v^\mu_{Hb}\;-\;\frac{6}{c}\,\Gamma^i_{\alpha\beta,\gamma}\,\Gamma^0_{\delta\mu}\,v^\mu_{Hb} \\ &-\frac{15}{c^2}\,\Gamma^0_{\alpha\beta}\,\Gamma^0_{\gamma\delta}\,\Gamma^i_{\mu\nu}\,v^\mu_{Hb}\,v^\nu_{Hb}\;+\;\frac{8}{c}\,\Gamma^0_{\alpha\mu}\,\Gamma^\mu_{\beta\gamma}\,\Gamma^i_{\delta\mu}\,v^\mu_{Hb} \\ &+\frac{12}{c}\,\Gamma^i_{\alpha\mu}\,\Gamma^\mu_{\beta\gamma}\,\Gamma^0_{\delta\nu}\,v^\nu_{Hb}\Big\}_b\,v^\alpha_{Hb}\,v^\beta_{Hb}\,v^\gamma_{Hb}\,v^\delta_{Hb}\;. \end{aligned} \tag{3–3e}$$

The index b means that all calculations have to be done at the position of the station b, which receives the signal as second. k_H^i is the spatial component of the **wave–vector** of a radio source H and v_H^i is the **3–coordinate–velocity** of an emmitted signal, which is the spatial component of the so–called **4–coordinate–velocity** v_H^α, that appears in (3–3c-e). The $\Gamma^\alpha_{\beta\gamma}$ are the **Christoffel–symbols**, partial derivatives of the metric coefficients $g_{\alpha\beta}$. P^α in (3–3c) pays attention to the influence of refraction.

For computation of the **theoretical delay** due to (3–3a) the radio source positions are needed in the geocentric spacetime Σ and the developed **stellar model** provides the necessary transformation from the catalog–spacetime $\hat{\Sigma}$ to the geocentric spacetime Σ. This **general–relativistic transformation of the source position** reads:

$$r_H^i \;=\; (e^i_k\,\tilde{e}^k_{\hat{j}}\,\hat{r}^j_H)\;+\;(\delta r^i)_1\;+\;(\delta r^i)_2\;+\;(\delta r^i)_3\;+\;(\delta r^i)_4 \tag{3–4a}$$

with

$$
\begin{aligned}
(\delta r^i)_1 &= \frac{1}{2}\, h_{00\,(2)}\, (e^i_{\bar{k}}\, \tilde{e}^k_{\hat{j}}\, \hat{r}^j_H) \;+\; \frac{1}{2}\, h_{im\,(2)}\, e^m_{\bar{k}}\, \tilde{e}^k_{\hat{j}}\, \hat{r}^j_H && \text{(3–4b)}\\
&\approx h_{00\,(2)}\, (e^i_{\bar{k}}\, \tilde{e}^k_{\hat{j}}\, \hat{r}^j_H) \\
(\delta r^i)_2 &= \frac{v^i}{c} \;-\; \frac{v^m}{c} \Big\{ \Big(\frac{1}{2}\frac{v^i}{c} \;+\; e^i_{\bar{k}}\, \tilde{e}^k_{\hat{j}}\, \hat{r}^j_H \Big) \, (e^m_{\bar{k}}\, \tilde{e}^k_{\hat{j}}\, \hat{r}^j_H) \Big\} \\
&\quad - \frac{1}{2}\frac{v^2}{c^2}\, (e^i_{\bar{k}}\, \tilde{e}^k_{\hat{j}}\, \hat{r}^j_H) && \text{(3–4c)}\\
(\delta r^i)_3 &= -\frac{1}{2}\, h_{0i\,(1)}\, h_{0m\,(1)}\, e^m_{\bar{k}}\, \tilde{e}^k_{\hat{j}}\, \hat{r}^j_H && \text{(3–4d)}\\
&\quad + (h_{0m\,(1)}\, e^m_{\bar{k}}\, \tilde{e}^k_{\hat{j}}\, \hat{r}^j_H)\, (e^i_{\bar{k}}\, \tilde{e}^k_{\hat{j}}\, \hat{r}^j_H) \\
(\delta r^i)_4 &= (h_{0m\,(1)}\, e^m_{\bar{k}}\, \tilde{e}^k_{\hat{j}}\, \hat{r}^j_H)\, \frac{v^i}{c} \;+\; \Big(h_{0m\,(1)}\, \frac{v^m}{c} \Big)\, (e^i_{\bar{k}}\, \tilde{e}^k_{\hat{j}}\, \hat{r}^j_H)\;. && \text{(3–4e)}
\end{aligned}
$$

$e^i_{\hat{j}}$ is the classical rotation matrix between the catalog–system and the geocentric system. $\tilde{e}^i_{\hat{j}}$ is an additional rotation matrix containing relativistic precession terms. The (δr^i) are corrections due to gravitation and relative motion of the two systems. As a matter of principle the supplements of the classical transformation are rather small and for this reason they are negligible except the aberration. A valuation of the terms is to find in *Börger (1998)*.

4 Numerical investigations

This section deals with data analysis using the new model, whereat these investigations mainly serve two aspects, that is to say the verification and classification of the quality of the geocentric relativistic model. For this purpose eleven experiments were analyzed and one batch solution was performed. Because four different relativistic models are available here four types of solution were created, but all with the same parametrization. With these given odds the quality of a solution is characterized first by the **WRMS of the observed delay**. Table 1 shows the results. The WRMS is the same

Experiment (database)	(3–3a) WRMS [ps]	HELLINGS WRMS [ps]	SHAPIRO WRMS [ps]	CONSENS WRMS [ps]
96MAR04XH	39,744	39,718	39,746	39,748
96APR15XH	49,869	49,854	49,855	49,873
96MAY20XH	64,061	64,088	64,062	64,015
96JUN10XH	44,746	44,822	44,743	44,742
96JUL15XH	61,512	61,556	61,487	61,509
96AUG12XH	60,844	60,954	60,871	60,878
96SEP11XH	58,970	59,015	58,935	58,966
96OCT21XH	51,358	51,417	51,412	51,357
96NOV20XH	32,459	32,410	32,402	32,440
97FEB10XH	31,548	31,581	31,532	31,535
97FEB20XH	23,504	23,506	23,502	23,475
batch solution	40,888	40,911	40,878	40,881

Tab. 1: WRMS of the observed delay

for all models. Only within the sub–picoseconds differences appear, but they do not have any level of

significance. Based on this first investigation one statement results: the new model is valid and well arranged in the order of accuracy as is easily recognized in the results of the batch solution.
To obtain further statements the parameter estimation of a single representative experiment is considered, i.e. station coordinates respectively baseline lengths are compared. For ease of comparison in table 2 the estimated baseline lengths are divided into a constant part and a rest. The formal errors

		(3–3a)		HELLINGS		SHAPIRO		CONSENS	
baseline	length	rest	Sigma	rest	Sigma	rest	Sigma	rest	Sigma
(name)	[mm]	[mm]		[mm]		[mm]		[mm]	
Fortaleza–HartRAO	7025279700,00	0,15	10,51	2,65	10,51	2,36	10,51	2,60	10,51
Fortaleza–Miami20	5402898100,00	38,03	13,04	39,01	13,03	39,47	13,03	39,72	13,04
Fortaleza–Westford	5897126000,00	94,36	8,17	95,98	8,17	95,99	8,17	96,18	8,17
Fortaleza–Wettzell	7215409500,00	15,42	10,30	18,14	10,30	17,54	10,30	17,94	10,30
HartRAO–Miami20	10814594300,00	58,89	27,11	61,27	27,10	62,20	27,10	62,10	27,11
HartRAO–Westford	10658658500,00	16,78	12,91	19,78	12,90	20,21	12,90	19,89	12,91
HartRAO–Wettzell	7832322500,00	6,12	8,74	8,62	8,74	8,68	8,74	8,53	8,74
Miami20–Westford	2044503900,00	80,67	9,30	80,96	9,30	81,16	9,30	81,10	9,30
Miami20–Wettzell	7588401900,00	91,37	18,17	93,06	18,17	93,55	18,17	93,54	18,17
Westford–Wettzell	5998325500,00	31,61	4,09	33,37	4,09	33,49	4,09	33,38	4,09

Tab. 2: Baseline lengths from 96APR15XH

are the same for all models and the estimated lengths of the conventional models are nearly the same for the conventional models. The results of the new model differ significantly (max. $2mm$). If these differences are interpreted as a result of a higher order modelling of the series expansions (3–3a) one may say that these differences represent a model refinement and should lead to slightly better results in the sense of being closer to reality. To show this the new geocentric model (3–3a) was applied in a reduced form, so that the resulting baselines are less close to reality than those resulting from the complete model. This reduced model, i.e. the neglecting of extensions and supplements, leads to results, that converge to the conventional ones. Of course further investigations have to be done to consolidate these findings.

5 References

Börger K (1996) , Determination of metric coefficients with VLBI, Proceedings of the 11th working meeting on european VLBI for Geodesy and Astrometry, Chalmers University of Technology, Research Report No. 177, Göteborg 1996

Börger K (1998) , Untersuchungen zur geozentrischen Modellbildung der relativistischen Langbasisinterferometrie, Mitteilungen aus den Geodätischen Instituten der Rheinischen Friedrich–Wilhelms–Universität Bonn, Nr. 84

Eubanks T M (editor) (1991) , A Consensus Model for Relativistic Effects in Geodetic VLBI, Proceedings of the USNO Workshop on Relativistic Models for Use in Space Geodesy, United States Naval Observatory, Washington